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Algebraic Symbols.

BY ARTHUR LATHAM BAKER.

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§1. $+$ is a sign of non-interference with the condition of the operand.

$-$ is a sign of interference to the extent of a complete reversal of direction, condition or properties, or whatever may be under consideration.

$\times$ is a sign of interference to the extent of doing to the operand whatever was done to unity to produce the operator.

$\div$ is a sign of interference with the conditions of the operand to the extent of doing to the operand whatever was done to the operator to produce unity: it is the inverse of $\times$.

$\sqrt{}$ is a sign of interference to the extent of introducing what may be called a *mean state* or *condition* between the operand and unity: such a condition that if we repeat on it the operation necessary to produce it from unity, the result will be the original operand, e. g. $\sqrt{a}$ is such a mean condition between a and 1 that if we perform upon $\sqrt{a}$ the same operation that we did on unity to produce $\sqrt{a}$, the result will be a. It breaks the operation a into two similar successive operations.

$\sqrt{-}$ is a sign of interference to the extent that a repetition of the interference will produce a reversal of conditions: it is the symbol denoting the *mean state* or *mean condition* between $+$ and $-$, a *mean reversor*.

$\sqrt{-}$ *as a factor* (*planar*).

§2. If we consider $+$ as a sign of non-interference with the primary direction of a line in a plane, and $-$ as a sign of reversal of its direction, $\sqrt{-}$ must be the symbol of such an interference or operation, the double application of which will produce $-$. The only interpretation of such an operation is the turning of the line through a right angle.

$\sqrt{-}$ is usually designated by i.

This gives us four symbols of directional operation:

$+$, which is neutral and inoperative;

$-$, which reverses;

i, which turns through a right angle positively;

$-i$, which turns through a right angle negatively.

This naturally leads to the Argand diagram and the theory of complex functions, not as a *convention*, but as a logical consequence.

i as an exponent (*planar*).

§3. Denote a vector (in a plane) whose tensor is m and whose amplitude is ϕ by

$$(m, \phi)^{+1}.$$

Then we will have

$$(m, \phi)^{-1} = \left(\frac{1}{m}, -\phi\right).$$

The change induced by the reversal of the exponent has changed m and ϕ respectively into $\frac{1}{m}$ and $-\phi$.

If now we can break up this one operation of reversal into two equal operations or sets of operations, we can interpret $(m, \phi)^i$, the *mean reversed* condition.

We will adopt the notation

e = the operation of taking the exponential in the natural system, e. g. ε^z.
e^{-1} = the operation of taking the anti-exponential or natural logarithm $= l$.
r = the reversal of an operation, as follows,
re^{-1} = the reversal of a logarithm or negative logarithm $= -e^{-1}$.
re = the reversal of the operation of taking the exponential $= e^{-1}$.
$r\phi$ = reversal of an angle $= -\phi$.
rm = reversal of a tensor $= m^{-1}$.

$\left(\frac{1}{m}, -\phi\right)$ can be written $(re^{-1}em, re^{-1}e\phi)$, the order of operations being indeterminate at present; or, omitting the operands and leaving simply the symbols of operation,

$$[re^{-1}e, re^{-1}e],$$

the brackets indicating that the enclosed quantities are symbols of operation upon the tensor and amplitude respectively.

Between $[1]$ and $[re^{-1}e]$ are three operations, performed upon two operands; six operations in all, which can be arranged in six different ways. If these six operations can be broken up into two equal sets of three each, one set of operations should define the operation i as an exponent.

We can break $re^{-1}e$ up into re^{-1} on the tensor and e on the amplitude, for instance, and in order to repeat the operation and yet introduce e^{-1} into the tensor and re into the amplitude, we must interchange the results. Indicating the interchange by the symbol $\overline{|\;\;|}\underline{|\;\;|}$, we have as the symbol of the whole set of operations

$$[\,\overline{|re^{-1}|}\underline{|\;e\;|}\,],$$

which would be read, perform re^{-1} on the tensor, e on the amplitude, and interchange the results.

Repeating the set of operations, we have

$$[\,\overline{|re^{-1}|}\underline{|\;e\;|}\,]^2 = [\,\overline{|re^{-1}|}\underline{|\;e\;|}\,](e^{\phi}, -lm) = \left(\frac{1}{m}, -\phi\right).$$

Thus the operation $[-1, -1]$ has been broken up into two equal sets of operations, viz.

$$[\,\overline{|re^{-1}|}\underline{|\;e\;|}\,]^2.$$

In a similar manner it can be broken up into

$$[\,\overline{|e^{-1}|}\underline{|\;re\;|}\,],\; [\,\overline{|\;e\;|}\underline{|re^{-1}|}\,],\; [\,\overline{|\;re\;|}\underline{|e^{-1}|}\,],\; [\,\overline{|\;r\;|}\underline{|ee^{-1}|}\,] \text{ and } [\,\overline{|ee^{-1}|}\underline{|\;r\;|}\,].$$

The last four are excluded for the reason that they do not give the correct result when applied to (m, ϕ). Thus l indicating the natural logarithm

$$[\,\overline{|\;e\;|}\underline{|re^{-1}|}\,]^2(m, \phi) = [\,\overline{|\;e\;|}\underline{|re^{-1}|}\,](-l\phi, e^m) = \left(-m, \frac{1}{\phi}\right),$$

$$[\,\overline{|\;re\;|}\underline{|e^{-1}|}\,]^2(m, \phi) = [\,\overline{|\;re\;|}\underline{|e^{-1}|}\,](l\phi, e^{-m}) = \left(-m, \frac{1}{\phi}\right),$$

$$[\,\overline{|\;r\;|}\underline{|ee^{-1}|}\,]^2(m, \phi) = [\,\overline{|\;r\;|}\underline{|ee^{-1}|}\,](\phi, m^{-1}) = (m^{-1}, \phi^{-1}).$$

$$[\,\overline{|ee^{-1}|}\underline{|\;r\;|}\,]^2(m, \phi) = [\,\overline{|ee^{-1}|}\underline{|\;r\;|}\,](-\phi, m) = (-m, -\phi),$$

This leaves us only two operations,

$$[\,\overline{|e^{-1}|}\underline{|\;re\;|}\,]^2(m, \phi) = [\,\overline{|e^{-1}|}\underline{|\;re\;|}\,](e^{-\phi}, lm) = (m^{-1}, -\phi), \ldots\ldots i,$$

$$[\,\overline{|re^{-1}|}\underline{|\;e\;|}\,]^2(m, \phi) = [\,\overline{|re^{-1}|}\underline{|\;e\;|}\,](e^{\phi}, -lm) = (m^{-1}, -\phi), \ldots\ldots -i.$$

These we designate by i and $-i$, the first resulting in a positive amplitude and the second in a negative.

§4. The same reasoning might be applied to the operation

$$[\,|\overline{E^{-1}}|\underline{\;rE\;}|\,],$$

where E is the exponential to any base and E^{-1} the corresponding logarithm, thus apparently giving another set of operations whose double application would result in $[-1, -1]$, and which apparently has the same right to be called i as has the former set.

But upon investigation we will find

$$[\,|\overline{E^{-1}}|\underline{\;rE\;}|\,] = [\,|\overline{\mu e^{-1}}|\underline{(re)^{\frac{1}{\mu}}}|\,],$$

where μ is the modulus of the system of logarithms corresponding to the base E.

This set of operations would be read, perform re upon the amplitude and raise to the $\frac{1}{\mu}$ power, e^{-1} on the tensor and multiply by μ, and interchange the results.

Obviously this operation differs from the one designated as i only by the introduction of the operator μ, and we are quite justified in considering i as the fundamental operation and the one above as a quasi multiple of i.

This assumption is also justified as resulting in the well-known formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

derived by other methods.

§5. Since

$$[\,|\overline{e^{-1}}|\underline{\;re\;}|\,](m,\ 0) = (m,\ 0)^{i} = (1,\ lm),$$

we get the theorem—*A real to the i power becomes a unit vector whose amplitude is the natural logarithm of the real.*

Since

$$[\,|\overline{e^{-1}}|\underline{\;re\;}|\,](1,\ \phi) = (1,\ \phi)^{i} = (e^{-\phi},\ 0),$$

we get the theorem—*A unit vector to the i power becomes a real whose tensor is the reciprocal of the exponential of the amplitude.*

Applied to $(e,\ 0)^{\phi}$ we get

$$[\,|\overline{e^{-1}}|\underline{\;re\;}|\,](e,\ 0)^{\phi} = [\,|\overline{e^{-1}}|\underline{\;re\;}|\,](e^{\phi},\ 0) = (1,\ \phi) = e^{i\phi},$$

or the well-known formula

$$e^{i\phi} = \cos\phi + i\sin\phi.$$

Applied to any vector, we get

$$(m\,,\,\phi)^i = (e^{-\phi},\ lm)\,.$$

We easily derive, where α and β are any vectors with tensors m and n, and amplitudes ϕ and θ,

$$\alpha^\beta = (m,\ \phi)^{(n,\ \theta)} = (m^{n\cos\theta}.e^{-\phi\sin\theta},\ \phi\cos\theta + lm^{n\sin\theta})\,.$$

Since $(e^\theta,\ 0)^i = e^{i\theta} = \cos\theta + i\sin\theta,$

$$(e^\theta,\ 0)^{ni} = (e^{i\theta})^n = (e^{\theta n},\ 0)^i = \cos n\theta + i\sin n\theta,$$

or $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (Demoivre's theorem).

§6. One equation,

$$e^{n\pi i} = \pm\, 1\,,$$

is noteworthy as containing in the most compact form possible all the algebraic symbols of operation: +, —, i, ×, ÷; the three what might be called universal constants, whose obtrusiveness is so marked in physical investigations, 1, π, e; and finally, as if to complete its comprehensiveness, a symbol for all integral numbers. The reason for this universal obtrusiveness of 1, π and e may consist in the unities which they represent: 1, the foundation of all calculation; π, the measure of geometric completeness, the measure of the completeness of the sweep of the horizon, or of the celestial dome; e, the measure of algebraic completeness, the measure of the involution of the smallest possible factor carried to its utmost limit, the measure of the completeness of the involution of the *only factor, the smallest possible improper fraction*, which will give a *definite result.* The complete involution of the *largest proper fraction* gives e^{-1}.

§7. From $e^{i\phi} = e^{i\phi + 2n\pi i} = (1,\ \phi)$ and $e^x = m$, we easily get

$$e^{x + i\phi + 2n\pi i} = m\,(1,\ \phi) = (m\,,\ \phi) = m\,(\cos\phi + i\sin\phi)$$

and

$$\begin{aligned} L(m,\ \phi) &= L(a + ib) = lm + i\phi + 2n\pi i,\\ L(m,\ 0) &= Lm = lm + 2n\pi i,\\ L(-\,m,\ 0) &= L(m,\ \pi) = L(-\,m) = lm + i\pi + 2n\pi i\\ &= lm + (2n + 1)\,\pi i, \end{aligned}$$

where L indicates the generalized logarithm and l the logarithm as ordinarily understood.

A special case in its simplest form is

$$L(-\,1) = i\pi.$$

§8. This interpretation of i as a symbol of *mean reversal* makes intelligible such equations as

$$(4.810475\ldots.)^i = \left(e^{\frac{\pi}{2}}\right)^i = i = \left(1, \frac{\pi}{2}\right),$$

$$(23.14\ldots.)^i \quad = (e^{\pi})^i = -1 = (1, \pi) = (-1, 0),$$

$$\log_{10} i \quad = i.0.68218\ldots.,$$

$$li = i\frac{\pi}{2} = i.1.570796\ldots.$$

i in Space.

§9. To give concreteness to our conceptions, let us take a geometrical illustration and see if it will lead us to consistent results when interpreted algebraically.

Naturally, a directed dimensionless magnitude is represented by a directed straight line, a vector. As naturally, a directionless magnitude, a scalar, would be represented by a spherical shell.

If a spherical shell of unit diameter represents $+1$, an adjacent unit shell would seem to represent -1, or the reversed state of the former.

The mean reversed state could apparently be found in two different ways:

A. By revolving the $+1$ sphere about the point of contact with the second through an angle of 90°, a repetition of which operation would produce the reversed state.

B. By shrinking the shell perpendicularly to the common line of centres in the proportion $\sin\theta$ (where $\theta = \cos^{-1}$ harmonic displacement of the shell element), and moving the resulting figure, a directed unit line, one-half its dimension toward the position of -1. A repetition of this operation would produce the -1 shell.

The operation A seems to be excluded by reason of its indefiniteness, leaving B as the *mean reversed* state of a scalar. This *mean reversed* state of a scalar is a *directed line in space*, a vector, which is definitely directed as soon as the scalar units $+1$, -1 are posited. We have also a quasi-physical reason why *a* VECTOR IN SPACE, *considered as a (multiplier) symbol of an algebraic operation to be performed upon another vector has the potency of a* MEAN REVERSOR. We naturally designate it therefore by the symbol i, or by some similar *mean reversor* symbol j, k, etc., each one being a directed $\sqrt{-1}$.

§10. *Multiplication of Parallel Vectors.*

Remembering that multiplication is the doing to the operand whatever was done to $+1$ to produce the operator, we see that ii must from the very conception of mean reversed state result in reversal or -1, or $ii = -1$.

Or, to carry out the geometric conception, we continue the shrinking of the vector in the ratio $\sin\theta$ ($\theta = \cos^{-1}$ harmonic displacement of the element) and move toward the position of -1 one-half its dimension in that direction, and the result is the -1 shell. Notice that since the factors are parallel, that is identical, we repeat precisely the same operation by which the operating factor was produced, that is, operation B. Hence

$$ii = -1,$$

or, *the repetition of a mean reversal results in total reversal*, or *the multiplication of parallel vectors gives a scalar or directionless magnitude which is however the reversal of the primary or undisturbed scalar unit from which the vectors sprang.*

§11. *Multiplication of Perpendicular Vectors.*

Since now the operating vector is perpendicular to the operand vector instead of parallel to it as in §10, the resulting mean reversed state must be as different from that of §10 as perpendicularity is different from parallelism, the extreme limit of difference, or anti-parallelism if I may so call it.

There are two ways of producing a *mean reversed* state of a vector:

C. By shrinking it in the ratio $\sin\theta$, etc., into a shell as in §10.

D. By revolving it 90°.

Case D was excluded from §10 by reason of its indefiniteness, and also by reason of its lack of similarity to the method by which the operating vector was produced.

Case C is excluded from this section by reason of its lack of anti-parallelism to the method by which the operating vector was produced.

This necessitates here the use of Case D, if it can be made definite in its result. It can be made definite from the fact that it must occupy such a position that the reversal of the operating vector shall return the result to its initial condition. This position is one of perpendicularity to both operator and operand, or

$$ij = k,$$

using i, j, k in the accepted notation of mutually perpendicular unit vectors. Hence, *The multiplication (symbolization of an operation to be performed) of one unit vector into another at right angles to it, turns the operand through a right angle in a plane perpendicular to the operator, to produce the mean reversed state.*

§12. *Multiplication of Inclined Vectors.*

Naturally the result would be a combination of those of §§10 and 11, that is, it would be partly scalar and partly vector, or

$$\alpha\beta = -\cos\theta + \varepsilon \sin\theta,$$

where α and β are two unit vectors inclined at θ° and ε is a unit vector perpendicular to α and β, since this formula satisfies both the limiting cases, §§10, 11.

Hence, *The multiplication (symbolization of an operation) of one unit vector into another inclined to it at an angle θ, thus producing the mean reversed state, turns the operand through a right angle into a plane perpendicular to the multiplier, makes its tensor sin θ and adds a scalar, $-$ cos θ.*

§13. To take a look at the subject from a different point of view; what, considered as the symbolization of an operation, is the potency of one vector in space over another?

Assuming that space is symmetrical, and consequently that no direction is pre-eminent, and that a change of sign in the multiplier (symbol of operation) or multiplicand reverses the product, we have

1°. The potency of a vector cannot be represented by +, for that would condemn us to impotency.

2°. It cannot be a — transformer, for the effect would be reversion whatever the relative inclination of the factors, which is unreasonable.

3°. If we consider it as a *mean reversor*, we get reconcilable results.

4°. If we consider it as a partial reversor more or less potent than a *mean* reversor, we get irreconcilable results as follows: If $\alpha\beta$ partially reverses β, then $-\alpha . -\beta$ should produce the same result, but this cannot be the case unless the first operation $\alpha\beta$ produces a mean state such that the second operation $-\alpha . -\beta$ can produce the same mean state.

Hence, a vector in space, considered as the symbol of an operation to be performed upon another vector, must be a mean reversor, and is appropriately represented by some directed $\sqrt{-1}$, viz. $\dot{i}$, $\dot{j}$, $\dot{k}$, etc.

A vector multiplier is a versor, specifically, a mean reversor.

§14. *Parallel Vector Factors.*

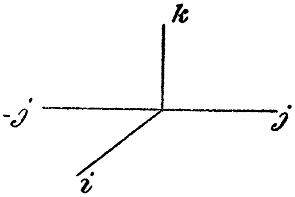
From the very nature of the operation, a mean reversal of a mean reversal must result in total reversal, or

$$\dot{i}\dot{i} = -1,$$

which agrees with §10.

§15. *Perpendicular Vector Factors.*

If $\dot{i}$ is to act as a mean reversor on $\dot{j}$, it must turn it to the position k, since $-\dot{i}$ acting on $-\dot{j}$ must produce the same result, which can only be when the mean state is as to position, perpendicular to both $\dot{i}$ and $\dot{j}$. Hence tentatively



$$\dot{i}\dot{j} = mk,$$

where m is the stretching factor, whatever it may be.

Operating again we have

$$\dot{i}.\dot{i}\dot{j} = \dot{i}.mk = m.\dot{i}k = m(-m\dot{j}) = -m^2\dot{j},$$

which to constitute reversal requires $m = 1$, and therefore

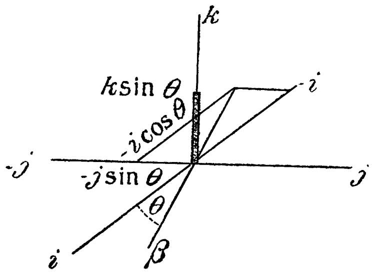
$$\dot{i}\dot{j} = k,$$

which agrees with §11.

§16. *Inclined Vector Factors.*

$$i\beta = \text{the mean reversed state of } \beta,$$

which, as before, must be as to direction some vector perpendicular to both of them, since $-i.-\beta$ must produce the same result. Hence tentatively



$$i\beta = sk,$$

where s is some scalar. Operating again with i to see if the second operation will produce reversal, we get, by §15,

$$i.i\beta = i.sk = s.ik = s(-j),$$

which is not reversal, but which would be if we could add $-ci$, afterwards multiplying by some scalar if necessary to elongate.

But this would require

$$i\beta = -c + sk,$$

since $$i.i\beta = -ic + i.sk = -ic - sj.$$

Evidently if $$-ic - js = -\beta,$$

then $$c = \cos\theta,\ s = \sin\theta,$$

and we have $$i\beta = -\cos\theta + k\sin\theta,$$

which agrees with §12.

§17. *Bivectors and Biquaternions.*

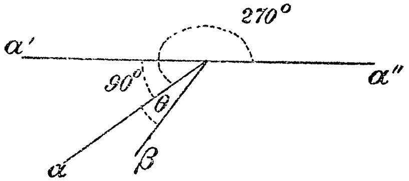
Apparently $$\sqrt{-1}\, i = -1,$$

and unit bivectors degenerate either into quaternions or a vector plus a scalar.

Using the geometrical conceptions of §9, and representing a quaternion by a vector combined with a scalar shell, we find that

$$\sqrt{-1}\,q = \sqrt{-1}\,(-\cos\theta + \varepsilon\sin\theta)$$

becomes $\sin\theta + \varepsilon\cos\theta$, or $-\sin\theta - \varepsilon\cos\theta$ according as we take the positive or negative mean reversed state. Hence, if $q = \alpha\beta$, $\sqrt{-1}\,q$ will be found by changing θ into $90 + \theta$ or $270 + \theta$.



That is, $\sqrt{-1}q$ *is found by revolving the operating factor* 90° (*or* 270°) *in the plane of the factors*, or if $q = \alpha\beta$, then $\sqrt{-1}\,q = \alpha'\beta$, $-\sqrt{-1}\,q = \alpha''\beta$.

§18. The significance of a multiplier should not be lost sight of; that it is strictly speaking not an operator with a potency of its own, but merely a symbolization of an operation to be performed, much as in $\log x$, log is not a multiplier but a symbol of operation. Looked at in this way, the property of non-commutativeness of factors is not new even to the ordinary mathematical operations, e. g. $\sqrt{a}$, $\tan a$, etc.

The symbolic character of the multiplier is apt to be lost sight of and the multiplication of one vector into another spoken of as if the multiplier had some quasi-mechanical power over the other vector.

The interpretation of i as a symbol of a mean reversed state makes the Argand diagram, the complex function and the quaternion analysis not conventions, but necessary and logical deductions from the algebraic symbols.

§19. As a symbol of algebraic operation dealing with dimensionless magnitudes, i seems to be necessarily represented, if at all, by a vector in space.

As a symbol of mean reversed state in a calculus where we have progressive and regressive multiplication, Grassmann's *Ausdehnungslehre* and allied branches, $\sqrt{-1}$ would almost necessarily be represented by a different concept or concepts.

§20. Does not this determine *the* meaning of $\sqrt{-1}$ in general, and also its method of representation in algebraic analysis? Similar methods of reasoning ought to give its method of representation in branches other than algebraic.

§21. In monodimensional domains $\sqrt{-}$ is purely imaginary, and the subject of operation is always a scalar. $(\sqrt{-})^{\infty} x = x$.

In bidimensional domains $(\sqrt{-})^{\infty} x = x + iy$ and the subject of operation is $w = f(x + iy)$. The condition of functionality is evidently the same as that for the symmetrical entrance of x and iy, viz.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial . iy},$$

whence we easily get the allied conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y},$$

the foundation of Complex Functions.

In tridimensional domains we have $(\sqrt{-})^{\infty} x = x + iy + jv + ku$, a quaternion.

Is this a hint that in the Calculus of Reals, Complex Functions and Quaternions, we have run the gamut of the Algebraic Calculi?

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